New Sequences from the Weber g Class Invariant

The purpose of this book has been the discussion of integer sequences. In my study of the g class invariants using the Ramanujan q-octic continued fraction I find that for every negative discriminant a real solution of the g class equations can be found. This real solution is an algebraic integer and is associated with other solutions, either real or complex to a polynomial. **Can the actual integer sequence associated with this polynomial be generated from the g class invariant and eventually from the real norm of the q-octic continued equation?**

We start with the value of q and τ determined from a discriminant, -n where n is an integer. The value of q is a complex number.

[1] $\tau = \frac{1+\sqrt{-d}}{4}$ with $q = e^{2\pi i \tau}$

From q one obtains the complex number for the q-octic continued fraction.

[2] $$u(\tau) = \sqrt{2} * q^{1/8} * \prod_{n>1} \frac{(1-q^{2n-1})}{(1-q^{4n-2})^2} = \cfrac{\sqrt{2}q^{1/8}}{1+\cfrac{q}{1+q+\cfrac{q^2}{1+q^2+\cfrac{q^3}{1+q^3+\cfrac{q^4}{1+q^4+\cfrac{q^5}{1+q^5 \ldots\ldots}}}}}}$$

The norm of the q-octic, CR2 is defined as its product with its conjugate.

[3] CR2 = $|u(\tau)| = u(\tau)$ * Conjugate $[u(\tau)]$

I have previously shown that the g class invariant is calculated from CR2.

[4] $g_{-n} = (2/|u(\tau)|)^{1/3}$

This equation is strictly true for odd discriminants and the definition of CR2 is slightly modified for even discriminants. I have previously shown that this modification is required to obtain the correct value of the g class invariant and the corresponding j-invariant. For even discriminants define two values of q and τ as follows,

[5] $t22 = \sqrt{n}/4$ with $q22 = e^{2\pi i \tau 22}$

[6] $t12 = \frac{1}{4}(1 + \sqrt{n/4})$ with $q12 = e^{2\pi i \tau 12}$

Use equation [2] to calculate $u(\tau 22)$ and $u(\tau 12)$ where $u(\tau 22)$ is real-valued and $u(\tau 12)$ is complex. The norm of $u(\tau 12) = |u(\tau 12)|$ is calculated as in [3] above and CR2 becomes,

[7] CR2(even) = $|u(\tau)| = \sqrt{2} * u(\tau 22)^3/|u(\tau 12)|$

The g class invariant for even discriminants is calculated as in [4].

Sequences have been found from the following discriminants, -D where D = (11,19, 43,67,163),(23,31), (55,100),(28,14,46,142),(34,82),(17,49,73,97, and 193). The grouping of discriminants indicates that different approaches are needed to find the corresponding integer sequence. Although following Weber's table and finding the real root that represents the characteristic equation will in many cases provide enough information for *Mathematica* to generate and solve for all the roots, my approach is to represent all roots, real and complex from the norm of the q-octic fraction. The relation of complex roots to the real root is not straightforward for all discriminants. In the cases of the above discriminants there is a direct relation if found between the real root(s) and the corresponding complex roots. The characteristic equations in these cases are third, sixth, or multiples of eighth order equations. Except for D= 23, and 31 all other class 3 discriminants are 9th order polynomials for which a simple formula was not found.

1. **Third order polynomial Solution**

There is an interesting form of solution for real third order (cubic) polynomials originally discussed by Gerolamo Cardano in his book *Ars Magma* published in 1545[1]. Cardano found that a reduced form of the cubic polynomial (after eliminating the x^2 monomial) could be solved for one real number from the equation. There are 3 solutions to the cubic polynomial and these solutions either have a form of one real and two complex or three real solutions. The reduced form of a general cubic

[8] $$x^3 - 3px - 2q = 0$$

Is solved from the rational numbers p and q.

[9] $$x = (q+w)^{1/3} + (q-w)^{1/3}$$

[10] $$w = (q^2 - p^3)^{1/2}$$

We consider when w is real $q^2 > p^3$. This constraint gives one real solution (x) and assures two complex solutions but also assures that the discriminant of the equation [8] and its unreduced form is negative! If $q^2 < p^3$ then even though the equation gives three real solutions, the discriminant is positive and ironically w is a complex number!

For general third order polynomials (including some of the Weber g class polynomials) the goal of reducing the integer sequence is simplified since the real solution obtained by Cardano's formula also produces the norm of the q-octic fraction directly by rearranging equation [4]!

It is important to point out how the Weber g class polynomials are directly obtained only from the negative integer discriminants from equations [1] and [2] above. The question remains regarding all other polynomials whether their characteristic equation is associated with a *rational* or *irrational* number d, in equation [1]. Can equation [2] for the q-octic fraction be extended to non-integer values?

We can take a closer look at integer sequences that are not from Weber g class polynomials. Consider the polynomial,

[11] $$x\hat{}3 - 5 * x\hat{}2 + 4 * x - 7$$

We remove the x^2 term by making the substitution y $-(x\hat{}2 + m * x)$ to find the resultant,

$$-49 - 28m - 35m^2 - 7m^3 + (-54 - m + 4m^2)y + (-17 - 5m)y^2 + y^3$$

where if m = -17/5 then equation [11] is reduced to

Roger Penrose, *Road to Reality*, Alfred A. Knopf, New York 2005. pp 75-76.

[12] $-\frac{10409}{125}-\frac{109y}{25}+y^3$

In Cardano form p = 109/75 and q = 10409/250.

As required, $q^2 > p^3$ and we can expect one real and two complex solutions to equation [12]. Using the formula [10] and [9] I find the real solution to [12] as y = 4.6990467967962780809. This can be verified as correct. Also, as expected the discriminant is negative although not an integer. Using *Mathematica*, $d = -\frac{2920196471}{15625} = -186892.5741$.

Researching a few other examples and comparing them to *Mathematica*, I find some interesting connections of the real solution to the two complex ones. First, in the reduced form the real par of the complex solution (a + i*b) can be found from the real solution, y as

[13] a = -y /2

Secondly, the product of the two complex solutions (a + i*b)* (a - i*b) = a^2 + b^2 is proportional to the cube root of the norm of the q-octic continued fraction! I have called this value CR2, and the proportional factor is the cube of the constant coefficient (2q in [12]) divided by 2. After simplification this product is shown to be,

[14] $a^2 - b^2 = 2*q/y$

Solving for the imaginary number b,

[15] $b = (a^2 - 2*q/y)^{1/2}$

This exercise demonstrates that real and complex solutions can be obtained for the reduced Cardano form of a cubic equation.

To find the solution to the original cubic equation [11] it is only necessary to solve a quadratic equation $y - (x^2 + m * x) = 0$ for x.

[16] $4.6990467967962780809 - (x^2 + m * x) = 0$

Two real solutions are found. It is not obvious at first which solution to use but since the discriminant of [11] is negative d = -2159 the positive root is preferred. It is also easy to obtain the complex roots of [11] ((a0 + i*b0)) by substituting the complex root (a + b) for y in [16]. One notices that the real coefficient of the complex solution of the unreduced equation is not easily related to the real solution as in equation [13]. A little guesswork is required to get this number.

If we create a sequence of integers, then this sequence is generated from powers of the sum of the roots of the cubic equation. The first term (n = 0) for the integer sequence is always 3 since the number to the zero power is always 1 for real and imaginary roots. The second term (n = 1) is the sum of all roots and since the complex roots are conjugates the b0 and -b0 terms cancel.

[17] xr +(a0 + i*b0)+ (a0 - i*b0) = xr + 2*a0 = some integer (I)

Since xr is found from [16], a0 = ((I) – xr)/2. There are several methods to check if the chosen integer is correct. First, as in the reduced case of equation [14] applies to equation [11]

[18] $a0^2 - b0^2 = c0/xr$

where c0 is the constant coefficient (7) of the unreduced equation. Define the variable as a function of the integer I as a0[I], b0[I] then b0[I] is,

[19] $b0[I] = (a0[I]^2 - 7/xr)^{1/2}$

[20] $a0[I] = (I - xr)/2$

Calculate b0[I] for integers from 0 to an upper integer (e.g 10). Then for some integer a0[I]+b0[I] is a solution to the unreduced equation and agrees with the quadratic solution mentioned above. Using this procedure guarantees that a sequence in terms of CR2 can be created. In the example above I = 5 and all roots since expressed as a function of xr can also be expressed as CR2:

[21] $xr = (2/CR2)^{1/3}$

Putting equations [19] to[21] the following sequence numbers for equation [11] are calculated.

[22] $$2^{-n}\left(\left(5-\sqrt{\left(-5+2^{1/3}\left(\frac{1}{CR2}\right)^{1/3}\right)^2-14 2^{2/3} CR2^{1/3}}-2^{1/3}\left(\frac{1}{CR2}\right)^{1/3}\right)^n+\left(5+\sqrt{\left(-5+2^{1/3}\left(\frac{1}{CR2}\right)^{1/3}\right)^2-14 2^{2/3} CR2^{1/3}}-2^{1/3}\left(\frac{1}{CR2}\right)^{1/3}\right)^n+2^{4n/3}\left(\frac{1}{CR2}\right)^{n/3}\right)$$

The first 10 numbers: {3,5,17,86,397,1760,7814,34809,155109,691007,3078262…}

Note that CR2 is the variable of choice since for the Weber g functions the real root is first obtained from the norm of the q octic continued fraction. I will show examples for various Weber g functions including polynomials of higher order than three. Many of the g functions do not have simple integer polynomials so the integer sequence cannot be found unless all the roots are identified.

Given a real root such y or xr above we can find the value of CR2. However, is it possible to find the value of -d associated with equation [1] and [2] above. One simple method of finding d uses Elliptic functions and the k invariants that I show to be calculated from CR2. We do not know whether a non-integer discriminant is odd or even so both equations for the k invariant can be used. These equations are found in Chapter 46 of this manuscript. In this example d = -130.261626… Using this value of d in equations [1] to [3] calculates CR2 = 0.0226224062064 which is close to the actual value by 1.5E[-10]. This illustrates that the "Ramanujan Ladder" continues beyond the integer discriminants and into a realm of rational and/or irrational numbers.

2. Examples from the Weber g functions.

In this example I will illustrate some of the steps for finding the integer sequence for d = -14. Weber points to a real solution $\sqrt{(\sqrt{2} * x)}$ where x is the solution of $x+1/x-(1+\sqrt{2}) = 0$. Is it reasonable to expect this solution results in an integer sequence? *Mathematica* can be used to find the answer. If equation [3] is used to find CR2 then *Mathematica* calculates a root approximation of an order 37 polynomial suggesting no possible integer sequence. However, when equation [7] is used for the even discriminant, an 8th order equation is revealed from the root approximation,

[23] $$4 - 16x^2 - 12x^4 - 8x^6 + x^8$$

providing hope that an integer solution exists. Are integer sequences possible for all Weber g functions with integer discriminants as well as for rational discriminants of other polynomials?

In this case of d = -14, CR2 = 0.4601629175285998931 and the real root value of g = 1.63194728431143552679573 6 is obtained from equation [4]. The even powers in equation [23] suggest that the square of g can reduce the equation to 4th order. For g^2 = 2.6632519387714693802878 1 is the root of

[24] $$4 - 8x + 6x^2 - 4x^3 + x^4$$

A 4th order equation with a real root can only have three more roots either all real or one real and 2 complex conjugates. Can the other real root be found from the value of CR2?

There is a structure and interactive framework to the roots of polynomials. It is worth using previous knowledge gained from the study of other polynomials. One structure uses CR2 to find the real root of a polynomial as in equation [21]. The value of CR2 can also be structured to give integer constants to a polynomial. In the above equation [24] the constant coefficient 4 (constant coefficient in [23) is important. Consider the value,

[25] $$g2 = (4*CR2)^{1/3}/2 = 0.7509616236016256685$$

This can be shown to be the second root of [24]. It is also the value of $2/g^2$. We only know this since we have a program *Mathematica* which shows that g^2 and g2 are roots of [24]. But could equation [25] be universal and provide the second solution to such polynomials? If so, we would not need to know the polynomial that discriminant -14 is associated with. The discriminant d of [24] is much larger: d = -7168 = $-(2^{10})*7$ so -14 is only a factor of d. It appears in this and the previous example that the polynomial discriminant is not a clear factor in deriving the root of the equation.

If we continue to consider the number 4 as important to this structure of [24] calculate the following equation similar to [20] to obtain a0 for the real part of the complex solution.

[26] $$a0 = (4 - g^2 - g2)/2 = 0.2928932188134524755 99$$

Then from [19] the similar structure for b0, the imaginary part of the complex root is,

[27] $$b0 = (a0^2 - 4/(g^2*g2))^{1/2} = 1.383551069665697251309i$$

Returning to the original 8th order equation [23], the solutions are {g,-g, $g2^{1/2}$, - $g2^{1/2}$, $(a0+b0)^{1/2}$, $(a0-b0)^{1/2}$, $-(a0+b0)^{1/2}$,- $(a0-b0)^{1/2}$}. Expressing the nth number in the sequence in terms of CR2

[28] $$\left(-\sqrt{\left(\left(2-\frac{(\frac{1}{CR2})^{2/3}}{2^{1/3}}-\frac{CR2^{2/3}}{2^{2/3}}\right)+\left(\sqrt{-2+\frac{1}{4}\left(-4+2^{2/3}(\frac{1}{CR2})^{2/3}+2^{1/3}CR2^{2/3}\right)^2}\right)\right)}\right)^n +$$

$$\sqrt{\left(\left(2-\frac{(\frac{1}{CR2})^{2/3}}{2^{1/3}}-\frac{CR2^{2/3}}{2^{2/3}}\right)+\left(\sqrt{-2+\frac{1}{4}\left(-4+2^{2/3}(\frac{1}{CR2})^{2/3}+2^{1/3}CR2^{2/3}\right)^2}\right)\right)}^{\,n} +$$

$$\left(-\sqrt{\left(\left(2-\frac{(\frac{1}{CR2})^{2/3}}{2^{1/3}}-\frac{CR2^{2/3}}{2^{2/3}}\right)-\left(\sqrt{-2+\frac{1}{4}\left(-4+2^{2/3}(\frac{1}{CR2})^{2/3}+2^{1/3}CR2^{2/3}\right)^2}\right)\right)}\right)^n +$$

$$\left(\sqrt{\left(\left(2-\frac{(\frac{1}{CR2})^{2/3}}{2^{1/3}}-\frac{CR2^{2/3}}{2^{2/3}}\right)-\left(\sqrt{-2+\frac{1}{4}\left(-4+2^{2/3}(\frac{1}{CR2})^{2/3}+2^{1/3}CR2^{2/3}\right)^2}\right)\right)}\right)^n +$$

$$\left(-\sqrt{(2^{1/3}*(CR2^{2/3}))}\right)^n + \left(\sqrt{(2^{1/3}*(CR2^{2/3}))}\right)^n + \left(-(2/CR2)^{1/3}\right)^n + \left((2/CR2)^{1/3}\right)^n$$

The first 20 numbers in [28]: {0, 8,0, 8,0, 32,0, 112,0, 288,0, 704,0, 1856,0, 5056,0, 13568,0, 35968}. This sequence is not found in OEIS.

Weber in his *Lehrbuch der Algebra* shows a solution in a similar form to d = -14 for two other larger even discriminants, d = -34 and d = -82. The analysis for these two sequences follows from similarity and structure. Values of the q-octic fraction are found for these even discriminants, CR2(34)= 0.20257152584593932 and CR2(82) = 0.05710324992223044158687. In both these examples a 16th order characteristic equation is found with *Mathematica*. The equation is easily reduced to 4th order by taking the 4th power of the root. Finding the 4 roots is straightforward however equation [26] is modified in the constant term to be 1/4th the coefficient of the monomial term in x. The coefficient (4) in [27] remains to be the constant coefficient of the characteristic equation. Both sequences can be expressed in one equation as functions of the corresponding CR2 value and a factor f1 = 1/4th the coefficient of the monomial term in x.

[29] $(-((f1/2 - 2^{1/3}(\frac{1}{CR2})^{4/3} - \frac{CR2^{4/3}}{2^{1/3}}) - (\sqrt{-4 + \frac{1}{4}(-f1 + 22^{1/3}(\frac{1}{CR2})^{4/3} + 2^{2/3}CR2^{4/3})^2}))^\wedge(1/4))^\wedge n + (((f1/2 - 2^{1/3}(\frac{1}{CR2})^{4/3} - \frac{CR2^{4/3}}{2^{1/3}}) - (\sqrt{-4 + \frac{1}{4}(-f1 + 22^{1/3}(\frac{1}{CR2})^{4/3} + 2^{2/3}CR2^{4/3})^2}))^\wedge(1/4))^\wedge n + (-((f1/2 - 2^{1/3}(\frac{1}{CR2})^{4/3} - \frac{CR2^{4/3}}{2^{1/3}}) + (\sqrt{-4 + \frac{1}{4}(-f1 + 22^{1/3}(\frac{1}{CR2})^{4/3} + 2^{2/3}CR2^{4/3})^2}))^\wedge(1/4))^\wedge n + (((f1/2 - 2^{1/3}(\frac{1}{CR2})^{4/3} - \frac{CR2^{4/3}}{2^{1/3}}) + (\sqrt{-4 + \frac{1}{4}(-f1 + 22^{1/3}(\frac{1}{CR2})^{4/3} + 2^{2/3}CR2^{4/3})^2}))^\wedge(1/4))^\wedge n + (i * (-((f1/2 - 2^{1/3}(\frac{1}{CR2})^{4/3} - \frac{CR2^{4/3}}{2^{1/3}}) + (\sqrt{-4 + \frac{1}{4}(-f1 + 22^{1/3}(\frac{1}{CR2})^{4/3} + 2^{2/3}CR2^{4/3})^2}))^\wedge(1/4)))^\wedge n + (i * (((f1/2 - 2^{1/3}(\frac{1}{CR2})^{4/3} - \frac{CR2^{4/3}}{2^{1/3}}) + (\sqrt{-4 + \frac{1}{4}(-f1 + 22^{1/3}(\frac{1}{CR2})^{4/3} + 2^{2/3}CR2^{4/3})^2}))^\wedge(1/4)))^\wedge n + (i * (-((f1/2 - 2^{1/3}(\frac{1}{CR2})^{4/3} - \frac{CR2^{4/3}}{2^{1/3}}) - (\sqrt{-4 + \frac{1}{4}(-f1 + 22^{1/3}(\frac{1}{CR2})^{4/3} + 2^{2/3}CR2^{4/3})^2}))^\wedge(1/4)))^\wedge n + (i * (((f1/2 - 2^{1/3}(\frac{1}{CR2})^{4/3} - \frac{CR2^{4/3}}{2^{1/3}}) - (\sqrt{-4 + \frac{1}{4}(-f1 + 22^{1/3}(\frac{1}{CR2})^{4/3} + 2^{2/3}CR2^{4/3})^2}))^\wedge(1/4)))^\wedge n + (i * (-(2^{2/3} * (CR2^{4/3}))^\wedge(1/4)))^\wedge n + (i * ((2^{2/3} * (CR2^{4/3}))^\wedge(1/4)))^\wedge n + (i * (-(2/CR2)^\wedge(1/3)))^\wedge n + (i * ((2/CR2)^\wedge(1/3)))^\wedge n + (-(2^{2/3} * (CR2^{4/3}))^\wedge(1/4))^\wedge n + ((2^{2/3} * (CR2^{4/3}))^\wedge(1/4))^\wedge n + (-(2/CR2)^\wedge(1/3))^\wedge n + ((2/CR2)^\wedge(1/3))^\wedge n$

For Calculation use CR2(34), f1 = 72 CR2(82), f1 = 456

The first 25 terms d =-34: {16, 0, 0, 0, 72, 0, 0, 0, 1808, 0,0,0, 38016, 0,0,0, 804928, 0,0,0, 17050752, 0,0,0, 361137152, .}

The first 21 terms d = -82 {16,0,0,0,456,0,0,0,52496,0,0,0,6019200,0,0,0,689756224,0,0,0,79041369216,. }

In these examples the constant factor f1 is found in the 4th entry of these sequences, (n = 4) but it is also the first integer after n= 0. It is also the monomial coefficient in x for the characteristic equation.

The last set of examples will calculate sequences for some odd discriminants of Weber functions that have cubic characteristic equations. These equations are 3rd order but unlike the first example the q-octic fraction is calculated to find the real root and not requiring the Cardano method. All these odd discriminants have a single real root and a set of conjugate complex roots. Like the second example, only the value of the norm of the q-octic fraction, CR2 and a constant ff are required for each discriminant. There is a difference in using CR2 to find the root of the Weber function instead of using the Cardano method. I find that the needed constant ff is found from the coefficient of the x^2 term instead of the x^0 term.

As an example, CR2 = 0.080360434160582 when d = -67. The g value is determined by *Mathematica* to be the solution of x^3-$2x^2$-2x-2. When d = -163, CR2 = 0.01329324764777368 and the g value is the root of the characteristic equation x^3 -$6x^2$+4x-2. Other cubic equations are found for d = -11,-19 and -43. I find that for the complex solution (a0 +/- ib0)

[30] a0 = (ff-g)/2

[31] $ib0 = (a0^2 - (4*CR2)^{1/3})^{1/2}$

where ff/-d is 6/(-163); 2/(-67); 0/(-19); 2/(-11); 2/(-43).

Based on all the cubic characteristic equations ff = coefficient of the x^2 monomial. If we revisit the first example where the characteristic equation was x^3 -5 x^2+4 x -7, I illustrated a method to find ff. It was found to be 5 which is the coefficient of the x^2 term of this equation!

[32] $$((2/\mathrm{CR2})^{1/3})^n + \left(\frac{1}{2}\left(\mathrm{ff} + \sqrt{(-\mathrm{ff} + 2^{1/3}(\frac{1}{\mathrm{CR2}})^{1/3})^2 - 4 2^{2/3}\mathrm{CR2}^{1/3}} - 2^{1/3}(\frac{1}{\mathrm{CR2}})^{1/3}\right)\right)^n + \left(\mathrm{ff}/2 - \frac{1}{2}\sqrt{(-\mathrm{ff} + 2^{1/3}(\frac{1}{\mathrm{CR2}})^{1/3})^2 - 4 2^{2/3}\mathrm{CR2}^{1/3}} - \frac{(\frac{1}{\mathrm{CR2}})^{1/3}}{2^{2/3}}\right)^n$$

Sequence d=-163: {3,6,28,150,800,4256,22636,120392,640320,3405624…}

This sequence is not found in OEIS.

Note: For d=-31, ff=1 and -23, ff=0 the g given by Weber's solutions are divided by $\sqrt{2}$. The correct solutions are obtained if g in [30] and $(4*CR2)^{1/3}$ in [31] are divided by $\sqrt{2}$.

Many other Weber g functions sequences can be found using the methods described above for g and CR2. Other odd discriminants where sequences are found based on CR2 are d = -17, -49, -73, -97 and -193. Some even sequences studied are d=-28, -46, -142. Sequences could also be found for d = -55 and d=-100. For other discriminants such as d=-91 and d=-71 two constants are required to obtain the sequence. CR2 gives half of the solution but a second constant CR2b is needed and not directly calculated from CR2. The product of these two constants is related to a third order equation but this cannot be determined unless a sixth order equation is solved for two of the real roots.

The physical world is described by mathematical principles. Within this mathematics are polynomials which describe sequences. We find this in many 2nd and 3rd order polynomials which exist in mathematics and their sequences are found in the physical and art world as Fibonacci and Perrin sequences. If all negative real numbers define a discriminant for an equation, then there exist infinite sequences in the mathematical world. How many of these sequences are yet to be found in the physical world and its universe? This book is only a beginning to a world with Integer Sequences.